Gravitational bound waveforms from amplitudes

Riccardo Gonzo

THE UNIVERSITY of EDINBURGH

ITMP seminars series

ITMP, 13 March 2024

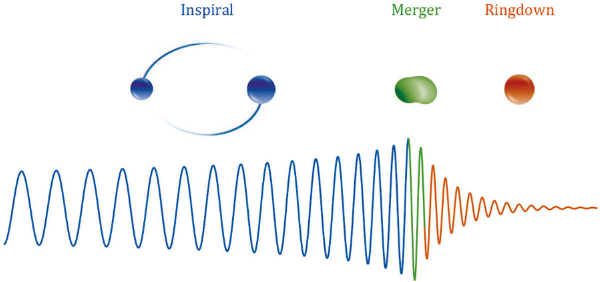
Content

Motivation and introduction (I)

- The recent discovery of gravitational waves calls for new analytical techniques to study the two-body problem.

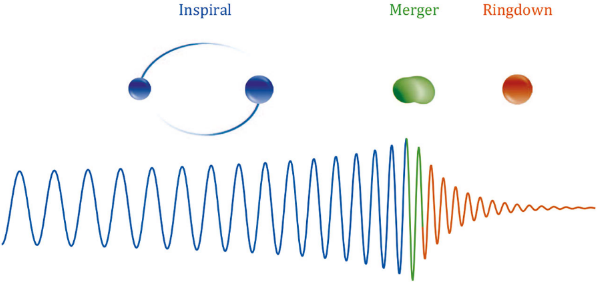
Motivation and introduction (I)

- The recent discovery of gravitational waves calls for new analytical techniques to study the two-body problem.
- We need waveform templates to extract the signal: the effective one-body (EOB) [Buonanno, Damour] and the self-force approach allow to combine analytical and numerical techniques for the evolution of compact binaries



Motivation and introduction (I)

- The recent discovery of gravitational waves calls for new analytical techniques to study the two-body problem.
- We need waveform templates to extract the signal: the effective one-body (EOB) [Buonanno, Damour] and the self-force approach allow to combine analytical and numerical techniques for the evolution of compact binaries



- Today: focus on the inspiral phase, where we can model compact objects as point particles in the spirit of effective field theory [Goldberger,Rothstein]

Motivation and introduction (II)

- Idea: use particle field theory tools ($\to$ scattering amplitudes)

Real world	EFT of point particles
Compact objects of mass M	Point particles of mass M
Spin effects of magnitude a	Spinning particles of classical spin a
Tidal effects, GR curvature corrections	Higher-dimensional operators
Absorption effects	Non-unitary absorption dofs

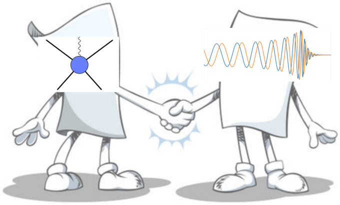
Motivation and introduction (II)

- Idea: use particle field theory tools ($\rightarrow$ scattering amplitudes)

Real world	EFT of point particles
Compact objects of mass M	Point particles of mass M
Spin effects of magnitude a	Spinning particles of classical spin a
Tidal effects, GR curvature corrections	Higher-dimensional operators
Absorption effects	Non-unitary absorption dofs

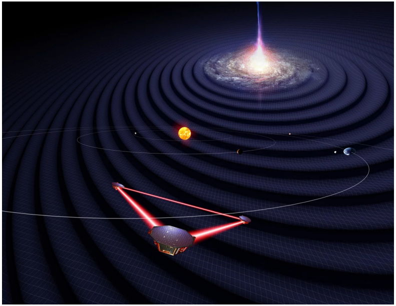
- Why amplitudes? (adapted to scattering orbits... bound orbits? Stay tuned!)

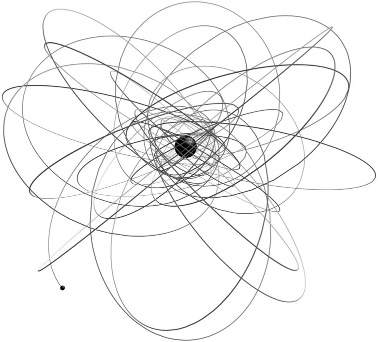
Amplitudes are gauge-invariant, universal objects which encode in a compact and analytic way the perturbative scattering dynamics for point particles in a QFT.
New perspective on GR!



Motivation and introduction (III)

- Analytic waveform templates are going to be necessary for extreme mass ratio inspirals, which are going to be detected by the LISA mission





As theoretical physicists, we need to work hard to be ready for 2035!

KMOC formalism for the two-body problem (I)

- Two-body scattering in GR: Consider as initial state two massive particles separated by an impact parameter b^μ [Kosower,Maybee,O'Connell=KMOC]

$$|\psi_{\text{in}}\rangle = \int \mathrm{d}\Phi\,(p_1, p_2)\,\psi_1(p_1)\psi_2(p_2)e^{i(b\cdot p_1)/\hbar}\,|p_1 p_2\rangle$$

with some wavefunctions ψ_1,ψ_2 localized on classical trajectories.

KMOC formalism for the two-body problem (I)

- Two-body scattering in GR: Consider as initial state two massive particles separated by an impact parameter b^μ [Kosower,Maybee,O'Connell=KMOC]

$$|\psi_{\text{in}}\rangle = \int \mathrm{d}\Phi\,(p_1, p_2)\,\psi_1(p_1)\psi_2(p_2)e^{i(b\cdot p_1)/\hbar}\,|p_1 p_2\rangle$$

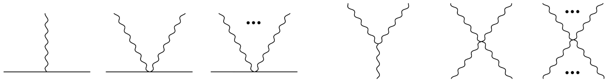
with some wavefunctions ψ_1,ψ_2 localized on classical trajectories.

- The dynamics of the evolution is determined by the action

$$S = -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}R + \sum_{j=1}^{2}\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi_j\partial_\nu\phi_j - m_j^2\phi_j^2\right) + S_{\mathrm{GF}}$$

where we perform the perturbative expansion

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\,, \quad \kappa = \sqrt{32\pi G_N} \to \text{Post-Minkowskian expansion in } G_N\,.$$



KMOC formalism for the two-body problem (II)

- We can compute classical observables $\mathcal{O}$ with expectation values

$$\langle\psi_{\text{in}}|\mathcal{S}^{\dagger}\mathcal{O}\mathcal{S}|\psi_{\text{in}}\rangle\Big|_{\hbar\to 0} = 2\Re i\langle\psi_{\text{in}}|\mathcal{O}T|\psi_{\text{in}}\rangle\Big|_{\hbar\to 0} + \langle\psi_{\text{in}}|T^{\dagger}\mathcal{O}T|\psi_{\text{in}}\rangle\Big|_{\hbar\to 0}$$

which the S-matrix $\mathcal{S} = 1 + iT$ gives both contributions linear in the amplitude T (and its conjugate $T^{\dagger}$) and quadratic ones $T^{\dagger}T$ (unitarity cuts).

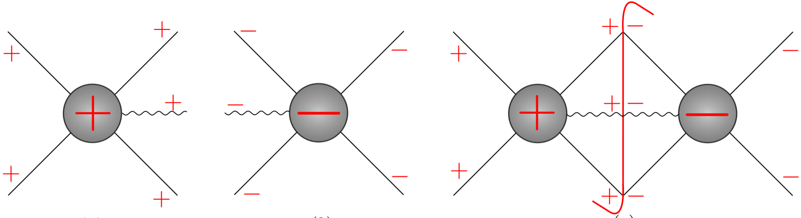
KMOC formalism for the two-body problem (II)

- We can compute classical observables $\mathcal{O}$ with expectation values

$$\langle\psi_{\text{in}}|\mathcal{S}^\dagger\mathcal{O}\mathcal{S}|\psi_{\text{in}}\rangle\Big|_{\hbar\to0} = 2\Re i\langle\psi_{\text{in}}|\mathcal{O}T|\psi_{\text{in}}\rangle\Big|_{\hbar\to0} + \langle\psi_{\text{in}}|T^\dagger\mathcal{O}T|\psi_{\text{in}}\rangle\Big|_{\hbar\to0}$$

  which the S-matrix $\mathcal{S} = 1 + iT$ gives both contributions linear in the amplitude T (and its conjugate $T^\dagger$) and quadratic ones $T^\dagger T$ (unitarity cuts).

- Connection with the "classical" on-shell reduction of the in-in approach in the $(+)/(-)$ basis [Britto, RG, Jehu]



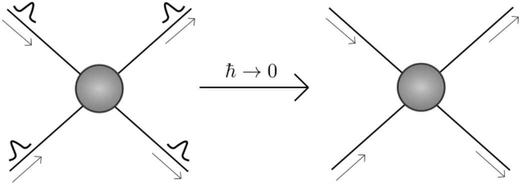
(a) (b) (c)

Classical limit from quantum field theory?

- How do we take the classical limit for the scattering of point particles?

Classical limit from quantum field theory?

- How do we take the classical limit for the scattering of point particles?
- Massive particles: use minimum-uncertainty wavefunctions localized on the classical trajectory [KMOC]



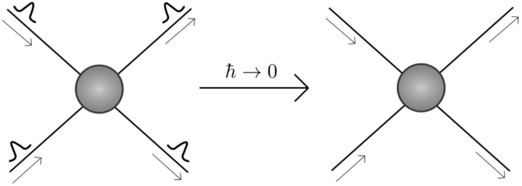
$$\psi(p) = \mathcal{N} m^{-1} \exp\left[-\frac{p \cdot u}{\hbar \ell_c / \ell_w^2}\right] \xrightarrow{\text{rest frame}} \mathcal{N}' \exp\left(-\frac{p^2}{2m^2 \ell_c^2 / \ell_w^2}\right)$$

where p^μ is the momentum, $\ell_{c,j} = \hbar / m_j$ is the Compton wavelength, ℓ_w the intrinsic spread of the wavefunction. If b^μ is the impact parameter we require,

$$\ell_{c,j} \ll \ell_w \ll b = \sqrt{-b^2}\,.$$

Classical limit from quantum field theory?

- How do we take the classical limit for the scattering of point particles?
- Massive particles: use minimum-uncertainty wavefunctions localized on the classical trajectory [KMOC]



$$\psi(p) = \mathcal{N} m^{-1} \exp\left[-\frac{p \cdot u}{\hbar \ell_c / \ell_w^2}\right] \xrightarrow{\text{rest frame}} \mathcal{N}' \exp\left(-\frac{p^2}{2m^2 \ell_c^2 / \ell_w^2}\right)$$

where p^μ is the momentum, $\ell_{c,j} = \hbar / m_j$ is the Compton wavelength, ℓ_w the intrinsic spread of the wavefunction. If b^μ is the impact parameter we require,

$$\ell_{c,j} \ll \ell_w \ll b = \sqrt{-b^2}\,.$$

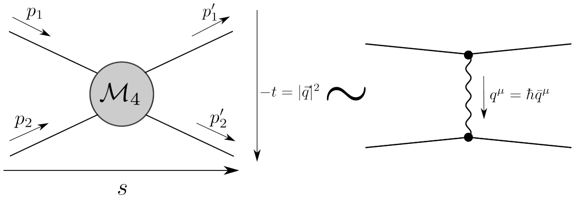
- Massless particles: use coherent states! [Cristofoli,RG,Kosower,O'Connell]

Classical limit of scattering amplitudes

- Conservative 4-pt amplitude $\mathcal{M}_4(p_1, p_2; p_1', p_2')$: in the classical limit $\hbar \to 0$

$$p_1^\mu := p_A^\mu + \hbar \frac{\bar{q}^\mu}{2}\,, \qquad (p_1')^\mu := p_A^\mu - \hbar \frac{\bar{q}^\mu}{2}\,, \qquad s = (p_A + p_B)^2\,,$$
$$p_2^\mu := p_B^\mu - \hbar \frac{\bar{q}^\mu}{2}\,, \qquad (p_2')^\mu := p_B^\mu + \hbar \frac{\bar{q}^\mu}{2}\,, \qquad t = -\hbar^2 |\vec{\bar{q}}|^2\,,$$

where p_A, p_B are the classical momenta and q is the momentum transfer.

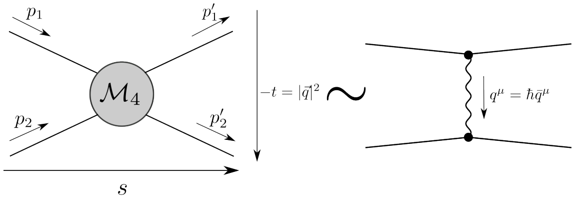


Classical limit of scattering amplitudes

- Conservative 4-pt amplitude $\mathcal{M}_4(p_1, p_2; p_1', p_2')$: in the classical limit $\hbar \to 0$

$$p_1^\mu := p_A^\mu + \hbar \frac{\bar{q}^\mu}{2}, \qquad (p_1')^\mu := p_A^\mu - \hbar \frac{\bar{q}^\mu}{2}, \qquad s = (p_A + p_B)^2,$$
$$p_2^\mu := p_B^\mu - \hbar \frac{\bar{q}^\mu}{2}, \qquad (p_2')^\mu := p_B^\mu + \hbar \frac{\bar{q}^\mu}{2}, \qquad t = -\hbar^2 |\vec{\bar{q}}|^2,$$

where p_A, p_B are the classical momenta and q is the momentum transfer.



- Generalization for the $4 + M$-pt amplitude $\mathcal{M}_{4+M}(p_1, p_2; p_1', p_2', k_1, \ldots, k_M)$

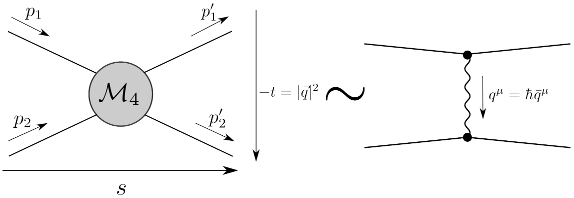
$$q_{1,2}^\mu = p_{1,2}^\mu - (p_{1,2}')^\mu = \hbar \bar{q}_{1,2}^\mu, \qquad k_j^\mu = \hbar \bar{k}_j^\mu, j = 1, \ldots, M.$$

Classical limit of scattering amplitudes

- Conservative 4-pt amplitude $\mathcal{M}_4(p_1, p_2; p_1', p_2')$: in the classical limit $\hbar \to 0$

$$p_1^\mu := p_A^\mu + \hbar\frac{\bar{q}^\mu}{2}\,, \qquad (p_1')^\mu := p_A^\mu - \hbar\frac{\bar{q}^\mu}{2}\,, \qquad s = (p_A + p_B)^2\,,$$
$$p_2^\mu := p_B^\mu - \hbar\frac{\bar{q}^\mu}{2}\,, \qquad (p_2')^\mu := p_B^\mu + \hbar\frac{\bar{q}^\mu}{2}\,, \qquad t = -\hbar^2|\vec{\bar{q}}|^2\,,$$

where p_A, p_B are the classical momenta and q is the momentum transfer.



- Generalization for the $4+M$-pt amplitude $\mathcal{M}_{4+M}(p_1, p_2; p_1', p_2', k_1, \ldots, k_M)$

$$q_{1,2}^\mu = p_{1,2}^\mu - (p_{1,2}')^\mu = \hbar\bar{q}_{1,2}^\mu\,, \qquad k_j^\mu = \hbar\bar{k}_j^\mu\,, j = 1, \ldots, M\,.$$

- Main lesson: only wavevectors $\bar{q}_{1,2}^\mu, \bar{k}_j$ are classical, need to restore $\hbar$!

Waveforms from KMOC formalism (I)

- How is the waveform derived from scattering amplitudes?

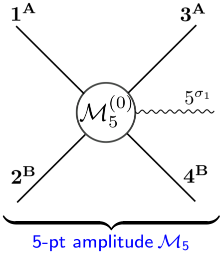
Waveforms from KMOC formalism (I)

- How is the waveform derived from scattering amplitudes?
- The on-shell expectation value of the time-domain waveform relevant for the inspiral phase is [Cristofoli,RG,Kosower,O'Connell]

$$\langle \psi_{\text{in}} | \mathcal{S}^\dagger h_{\mu\nu}(x) \mathcal{S} | \psi_{\text{in}} \rangle = \frac{1}{\hbar^{\frac{1}{2}}} 2\Re \sum_{\sigma=\pm} \int \mathrm{d}\Phi(k)\, \varepsilon_\mu^{*(\sigma)}(k) \varepsilon_\nu^{*(\sigma)}(k) \tilde{j}(b; k^\sigma) e^{-ik\cdot x/\hbar}$$

where at leading Post-Minkowskian order only the 5-pt amplitude is relevant

$$\tilde{j}(b; k^{\sigma_1}) \equiv \int \mathrm{d}\Phi(p_1' p_2' p_1 p_2) \psi^*(p_1', p_2') \psi(p_1, p_2) e^{-ib\cdot \bar{q}_1}$$



Waveforms from KMOC formalism (II)

- Assuming that the measurement distance is much larger than the impact parameter, so that there is a unique and well-defined direction,

$$G_{\text{ret}}(x) = i\theta\left(x^0\right)\int \mathrm{d}\Phi(k)\left(e^{-ik\cdot x} - e^{ik\cdot x}\right) = \frac{1}{4\pi|\vec{x}|}\delta\left(x^0 - |\vec{x}|\right)$$

Waveforms from KMOC formalism (II)

- Assuming that the measurement distance is much larger than the impact parameter, so that there is a unique and well-defined direction,

$$G_{\text{ret}}(x) = i\theta\left(x^0\right)\int \mathrm{d}\Phi(k)\left(e^{-ik\cdot x} - e^{ik\cdot x}\right) = \frac{1}{4\pi|\vec{x}|}\delta\left(x^0 - |\vec{x}|\right)$$

- we get for the strain at $x^0 > 0$ [Cristofoli,RG,Kosower,O'Connell]

$$h(x) = \frac{\kappa}{8\pi|\vec{x}|}\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\left[\tilde{j}(b;k^-)e^{-i\omega u} + \tilde{j}(b;k^+)^* e^{i\omega u}\right],$$

$$\tilde{j}(b;k^h) = \frac{1}{(2\pi)^2}\int \underbrace{\left[\prod_{i=1,2}\mathrm{d}^4\bar{q}_i\delta\left(2p_i\cdot\bar{q}_i\right)\right]}_{\text{Measure } \mathrm{d}\mu} e^{i(\bar{q}_1\cdot b_1 + \bar{q}_2\cdot b_2)}\underbrace{\mathcal{M}^{(0)}_{5,\mathrm{cl}}\left(\bar{q}_1,\bar{q}_2,\bar{k}^h\right)}_{\propto\delta^4(q_1+q_2-k)},$$

where $u = x^0 - |\vec{x}|$ is the retarded time.

Waveforms from KMOC formalism (II)

- Assuming that the measurement distance is much larger than the impact parameter, so that there is a unique and well-defined direction,

$$G_{\rm ret}(x) = i\theta\left(x^0\right)\int \mathrm{d}\Phi(k)\left(e^{-ik\cdot x} - e^{ik\cdot x}\right) = \frac{1}{4\pi|\vec{x}|}\delta\left(x^0 - |\vec{x}|\right)$$

- we get for the strain at $x^0 > 0$ [Cristofoli,RG,Kosower,O'Connell]

$$h(x) = \frac{\kappa}{8\pi|\vec{x}|}\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\left[\tilde{j}(b;k^-)e^{-i\omega u} + \tilde{j}(b;k^+)^* e^{i\omega u}\right],$$

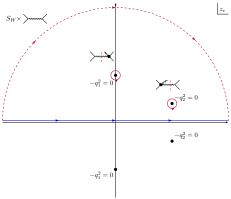
$$\tilde{j}(b;k^h) = \frac{1}{(2\pi)^2}\int \underbrace{\left[\prod_{i=1,2}\mathrm{d}^4\bar{q}_i\delta\left(2p_i\cdot\bar{q}_i\right)\right]}_{\text{Measure } \mathrm{d}\mu} e^{i(\bar{q}_1\cdot b_1 + \bar{q}_2\cdot b_2)}\underbrace{\mathcal{M}^{(0)}_{5,\rm cl}\left(\bar{q}_1,\bar{q}_2,\bar{k}^h\right)}_{\propto\delta^4(q_1+q_2-k)},$$

  where $u = x^0 - |\vec{x}|$ is the retarded time.
- We can now start to compute scattering waveforms!

Tree-level waveform for Schwarzschild black holes (I)

- Use on-shell tools:

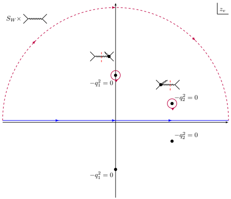
  Simplify the phase space integration of the 5-pt amplitude using S-matrix analyticity and unitarity (factorization into 3-pt and 4-pt amplitudes)



Tree-level waveform for Schwarzschild black holes (I)

- Use on-shell tools:

  Simplify the phase space integration of the 5-pt amplitude using S-matrix analyticity and unitarity (factorization into 3-pt and 4-pt amplitudes)



- Result: new compact representation of the tree-level waveform! [Kovacs,Thorne; Jakobsen,Mogull,Plefka,Steinhoff; De Angelis,RG,Novichkov]

$$h^{(0)}(x) = \frac{G_N^2 m_1 m_2}{|\vec{x}|\sqrt{-b^2}} \frac{1}{\bar{w}_1^2 \bar{w}_2^2 \sqrt{1+T_2^2}\left(\gamma + \sqrt{\left(1+T_1^2\right)\left(1+T_2^2\right)} + T_1 T_2\right)}$$

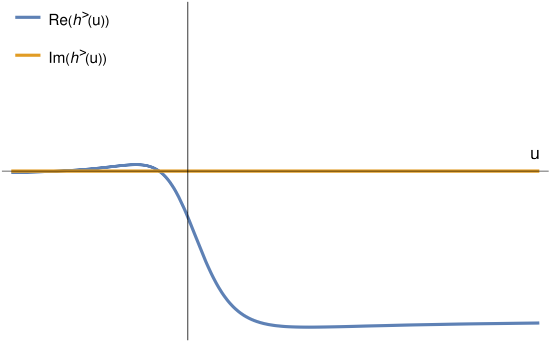
$$\times \left(\frac{3\bar{w}_1 + 2\gamma\left(2T_1 T_2 \bar{w}_1 - T_2^2 \bar{w}_2 + \bar{w}_2\right) - \left(2\gamma^2-1\right)\bar{w}_1}{\gamma^2-1} f_{1,2}^2 \right.$$

$$- \frac{4\gamma T_2 \bar{w}_2 f_1 + 2\left(2\gamma^2-1\right)\left[T_1\left(1+T_2^2\right)\bar{w}_2 f_1 + T_2(T_1 T_2 \bar{w}_1 + \bar{w}_2) f_2\right]}{\sqrt{\gamma^2-1}} f_{1,2}$$

$$\left. + 4\left(1+T_2^2\right)\bar{w}_2 f_1 f_2 - 4\gamma\left(1+T_2^2\right)\bar{w}_2\left(f_1^2+f_2^2\right) + 2\left(2\gamma^2-1\right)\left(1+2T_2^2\right)\bar{w}_2 f_1 f_2 \right) + (1 \leftrightarrow 2)$$

Tree-level waveform for Schwarzschild black holes (II)

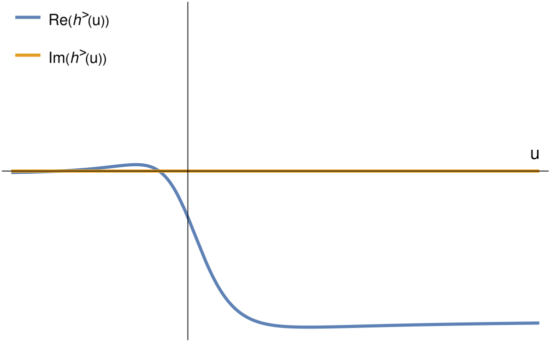
- The tree-level scattering waveform in the equatorial plane looks like



Most of the energy is released during the closest approach ($\sim$ periastron)!

Tree-level waveform for Schwarzschild black holes (II)

- The tree-level scattering waveform in the equatorial plane looks like



Most of the energy is released during the closest approach ($\sim$ periastron)!

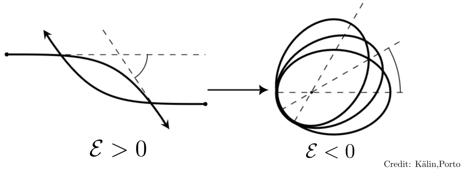
- Very different compared to (quasi)-periodic bound waveforms for compact binaries. . . is it possible to establish a connection?

From scattering to bound dynamics

- Classical scattering amplitudes describe hyperbolic encounters. If we define

$$\mathcal{E} := \frac{E - m_1 - m_2}{\mu}, \qquad p_\infty^2 = -\tilde{p}_\infty^2 = \frac{E^2 - (m_1 + m_2)^2}{2 m_1 m_2},$$

we have $\mathcal{E}, p_\infty^2 > 0$ for scattering orbits and $\mathcal{E}, p_\infty^2 < 0$ for bound orbits.

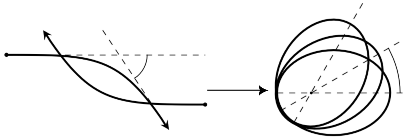


From scattering to bound dynamics

- Classical scattering amplitudes describe hyperbolic encounters. If we define

$$\mathcal{E} := \frac{E - m_1 - m_2}{\mu}\,, \qquad p_\infty^2 = -\tilde{p}_\infty^2 = \frac{E^2 - (m_1 + m_2)^2}{2 m_1 m_2}\,,$$

we have $\mathcal{E}, p_\infty^2 > 0$ for scattering orbits and $\mathcal{E}, p_\infty^2 < 0$ for bound orbits.



- Two powerful methods to extract bound state physics from amplitudes:
1) Extract perturbatively the classical potential ($\sim$ Hamiltonian) valid for arbitrary orbits [Niell,Rothstein;Cheung,Rothstein,Solon]
2) Gauge invariant map between scattering and bound observables [Kälin,Porto]

$$\boxed{\mathcal{O}^{>}(\mathcal{E} > 0, J, c_X, a_1, a_2, m_1, m_2) \to \mathcal{O}^{<}(\mathcal{E} < 0, J, c_X, a_1, a_2, m_1, m_2)}$$

which can be derived from the Bethe-Salpeter eq. [Adamo,RG; Adamo,RG,Ilderton].

The bound state equation in quantum mechanics (I)

- How can we describe bound states of point particles? Start with the probe limit in a linearized Schwarzschild background

$$\left[\hbar^2 \boldsymbol{\nabla}^2 + |\mathbf{p}|^2 + \frac{2\hbar|\mathbf{p}|\zeta}{r}\right] \bar{\Psi}(\mathbf{x}) = 0\,, \quad \zeta := \frac{G_N m_B}{\hbar} \frac{(2E^2 - m_A^2)}{\sqrt{E^2 - m_A^2}}\,,$$

which can be mapped into the (solvable) Coulomb potential [Kabat,Ortiz].

The bound state equation in quantum mechanics (I)

- How can we describe bound states of point particles? Start with the probe limit in a linearized Schwarzschild background

$$\left[\hbar^2 \boldsymbol{\nabla}^2 + |\mathbf{p}|^2 + \frac{2\hbar|\mathbf{p}|\zeta}{r}\right] \bar{\Psi}(\mathbf{x}) = 0\,, \quad \zeta := \frac{G_N m_B}{\hbar} \frac{(2E^2 - m_A^2)}{\sqrt{E^2 - m_A^2}}\,,$$

  which can be mapped into the (solvable) Coulomb potential [Kabat,Ortiz].
- We are familiar to the eigenvalue problem

$$\hat{H}\left|\bar{\Psi}\right\rangle = E\left|\bar{\Psi}\right\rangle, \qquad \hat{H} = \hbar^2 \boldsymbol{\nabla}^2 + |\mathbf{p}|^2 + V, \qquad V(r) \propto \frac{G_N}{r}\,,$$

  which can be solved exactly (at all orders in the coupling G_N)

$E > m_A \leftrightarrow$ scattering plane wave $\bar{\Psi}_{\mathbf{p}} \propto e^{i\mathbf{p}^{>}\cdot\vec{x}}$ $\leftrightarrow$ continuous spectrum $E_{\mathbf{p}}$

$E < m_A \leftrightarrow$ normalizable wavefunction $\bar{\Psi}_n \propto e^{-E_n|\vec{x}|}$ $\leftrightarrow$ discrete spectrum E_n

  where $>$ (resp. $<$) stands for scattering orbits (resp. bound orbits).

The bound state equation in quantum mechanics (I)

- How can we describe bound states of point particles? Start with the probe limit in a linearized Schwarzschild background

$$\left[\hbar^2 \boldsymbol{\nabla}^2 + |\mathbf{p}|^2 + \frac{2\hbar|\mathbf{p}|\zeta}{r}\right]\bar{\Psi}(\mathbf{x}) = 0\,, \quad \zeta := \frac{G_N m_B}{\hbar}\frac{(2E^2 - m_A^2)}{\sqrt{E^2 - m_A^2}}\,,$$

  which can be mapped into the (solvable) Coulomb potential [Kabat,Ortiz].
- We are familiar to the eigenvalue problem

$$\hat{H}\left|\bar{\Psi}\right\rangle = E\left|\bar{\Psi}\right\rangle\,, \qquad \hat{H} = \hbar^2\boldsymbol{\nabla}^2 + |\mathbf{p}|^2 + V\,, \qquad V(r) \propto \frac{G_N}{r}\,,$$

  which can be solved exactly (at all orders in the coupling G_N)

$$E > m_A \leftrightarrow \text{scattering plane wave } \bar{\Psi}_{\mathbf{p}} \propto e^{i\mathbf{p}^{>}\cdot\vec{x}} \quad \leftrightarrow \text{continuous spectrum } E_{\mathbf{p}}$$

$$E < m_A \leftrightarrow \text{normalizable wavefunction } \bar{\Psi}_n \propto e^{-E_n|\vec{x}|} \leftrightarrow \text{discrete spectrum } E_n$$

  where $>$ (resp. $<$) stands for scattering orbits (resp. bound orbits).
- Does a relation exist between scattering and bound wavefunctions?

The bound state equation in quantum mechanics (II)

- We find that the scattering and bound wavefunction [Messiah, Gottfried]

$$\bar{\Psi}^{>}_{\mathbf{p}}(x) = e^{\pi\zeta/2}\,\Gamma(1-i\zeta)\,{}_1F_1\!\left(i\zeta;1;\frac{i(|\mathbf{p}|r-\mathbf{p}\cdot\mathbf{r})}{\hbar}\right)e^{-ip\cdot x/\hbar}\,,$$

$$\bar{\Psi}^{<}_{n\ell m}(x) = e^{-iE_n t/\hbar}\,R^{<}_{n\ell}(r)\,Y_{\ell m}(\theta,\phi)\,,\quad \text{Quantization: } i\zeta = n\,,$$

have a simple relation in partial wave basis [Adamo, RG, Ilderton; Gottfried]

$$\bar{\Psi}^{<}_{n\ell m}(x,\sqrt{1-y^2}) = \mathrm{Res}_{\zeta=-in}\left[\bar{\Psi}^{>}_{\ell m}(x,\sqrt{y^2-1}\to +i\sqrt{1-y^2})\right]\,.$$

with a single branch cut prescription in $y = E/m_A$ [Adamo, RG].

The bound state equation in quantum mechanics (II)

- We find that the scattering and bound wavefunction [Messiah, Gottfried]

$$\bar{\Psi}^{>}_{\mathbf{p}}(x) = e^{\pi\zeta/2}\,\Gamma(1-i\zeta)\,{}_1F_1\!\left(i\zeta;1;\frac{i(|\mathbf{p}|r-\mathbf{p}\cdot\mathbf{r})}{\hbar}\right)e^{-ip\cdot x/\hbar}\,,$$

$$\bar{\Psi}^{<}_{n\ell m}(x) = e^{-iE_n t/\hbar}\,R^{<}_{n\ell}(r)\,Y_{\ell m}(\theta,\phi)\,,\quad \text{Quantization: } i\zeta = n\,,$$

have a simple relation in partial wave basis [Adamo, RG, Ilderton; Gottfried]

$$\bar{\Psi}^{<}_{n\ell m}(x,\sqrt{1-y^2}) = \mathrm{Res}_{\zeta=-in}\left[\bar{\Psi}^{>}_{\ell m}(x,\sqrt{y^2-1}\to +i\sqrt{1-y^2})\right]\,.$$

with a single branch cut prescription in $y = E/m_A$ [Adamo, RG].

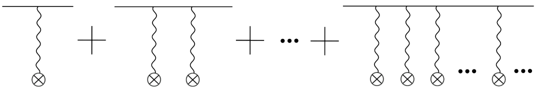
- Useful reformulation in terms of $p_\infty = \sqrt{y^2-1}$ and $\tilde{p}_\infty = \sqrt{1-y^2}$

$$\bar{\Psi}^{<}_{n\ell m}(x,\tilde{p}_\infty) = \mathrm{Res}_{\zeta=-in}\left[\bar{\Psi}^{>}_{\ell m}(x,p_\infty = +i\tilde{p}_\infty)\right]$$

The residue comes from the the bound state pole ($\sim \Gamma(1-i\zeta)$) in the amplitude $\bar{\Psi}^{>}_{\mathbf{p}}$: can we simplify the map?

The bound state equation in quantum mechanics (III)

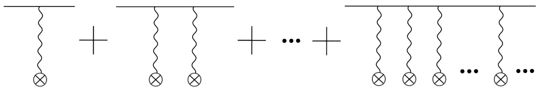
- In perturbation theory, the bound state energy pole is generated by the iteration of the potential $V + VGV + \cdots + V(GV)^n$:



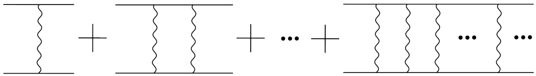
so in some sense only V should be relevant!

The bound state equation in quantum mechanics (III)

- In perturbation theory, the bound state energy pole is generated by the iteration of the potential $V + VGV + \cdots + V(GV)^n$:

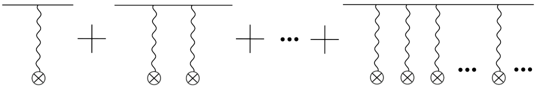


  so in some sense only V should be relevant!
- The natural generalization of the previous picture to the non-relativistic (Newtonian) two-body problem is given by the "ladder approximation"



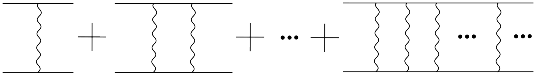
The bound state equation in quantum mechanics (III)

- In perturbation theory, the bound state energy pole is generated by the iteration of the potential $V + VGV + \cdots + V(GV)^n$:

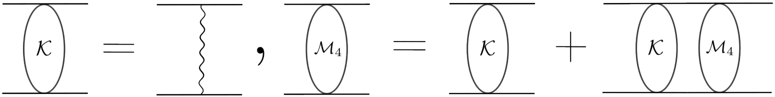


so in some sense only V should be relevant!

- The natural generalization of the previous picture to the non-relativistic (Newtonian) two-body problem is given by the "ladder approximation"



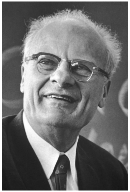
We can write it as an amplitude recursion relation



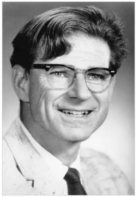
which is nothing else that the (quantum) Bethe-Salpeter equation!

The bound state equation in quantum field theory

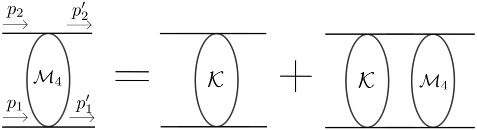
- The Bethe-Salpeter equation is a recursion relation for 4-pt amplitudes, which generate the bound energy poles via the iteration of a two-massive particle irreducible kernel (2MPI) $\mathcal{K}$



Bethe-



Salpeter



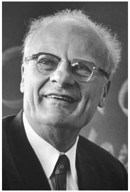
equation

$$\mathcal{M}_4(p_1, p_2; p_1', p_2') = \mathcal{K}(p_1, p_2; p_1', p_2') + \int \hat{\mathrm{d}}^4 s_1 \, \mathcal{K}(p_1, p_2; s_1, s_2)\Delta(s_1, s_2)\mathcal{M}_4(s_1, s_2; p_1', p_2') \, ,$$

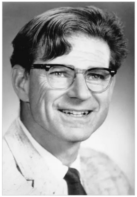
where $\Delta(s_1, s_2)$ is the two-body propagator.

The bound state equation in quantum field theory

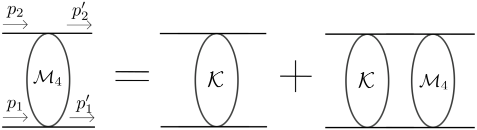
- The Bethe-Salpeter equation is a recursion relation for 4-pt amplitudes, which generate the bound energy poles via the iteration of a two-massive particle irreducible kernel (2MPI) $\mathcal{K}$



Bethe-



Salpeter



equation

$$\mathcal{M}_4(p_1, p_2; p_1', p_2') = \mathcal{K}(p_1, p_2; p_1', p_2') + \int \hat{\mathrm{d}}^4 s_1\, \mathcal{K}(p_1, p_2; s_1, s_2)\Delta(s_1, s_2)\mathcal{M}_4(s_1, s_2; p_1', p_2')\,,$$

where $\Delta(s_1, s_2)$ is the two-body propagator.

- What is the classical limit of this recursion relation?

The classical Bethe-Salpeter equation

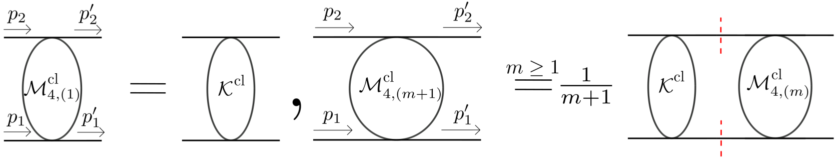
- We obtain the classical Bethe-Salpeter equation from quotienting diagrams by symmetrization over internal graviton exchanges: [Adamo, RG]

$$\mathcal{M}^{\mathrm{cl}}_{4,(m+1)}(p_A, p_B, q) = \begin{cases} \mathcal{K}_{\mathrm{cl}}(p_A, p_B, q) & \text{if } m = 0 \\ \frac{1}{m+1} \int \hat{\mathrm{d}}^4 l \, \mathcal{K}_{\mathrm{cl}}(p_A, p_B, l) G_{\mathrm{cl}}(p_A, p_B, l) \mathcal{M}^{\mathrm{cl}}_{4,(m)}(p_A, p_B, q - l) & \text{if } m \geq 1 \end{cases} .$$

where the two-body propagator is replaced by its on-shell version

$$G_{\mathrm{cl}}(p_A, p_B, l) = \hat{\delta}(2l \cdot p_A) \hat{\delta}(2l \cdot p_B) \, ,$$

and (m) is the number of classical two-massive particle irreducible diagrams.



Exponentiation of the classical kernel: an exact solution

- Going to impact parameter space (i.e. to the partial wave basis)

$$\widetilde{f}(b) \equiv \int \hat{d}^4 q \hat{\delta}(2p_A \cdot q)\, \hat{\delta}(2p_B \cdot q)\, e^{i(q\cdot b)/\hbar} f(q)\,,$$

the classical BSE becomes

$$\widetilde{\mathcal{M}}^{\text{cl}}_{4,(m+1)}(p_A, p_B, b) = \begin{cases} \widetilde{\mathcal{K}}_{\text{cl}}(p_A, p_B, b) & \text{if } m = 0 \\ \frac{1}{m+1}\widetilde{\mathcal{K}}_{\text{cl}}(p_A, p_B, b)\widetilde{\mathcal{M}}^{\text{cl}}_{4,(m)}(p_A, p_B, b) & \text{if } m \geq 1 \end{cases},$$

which means that the final solution exponentiates exactly [Adamo,RG]

$$\widetilde{\mathcal{M}}^{\text{cl}}_{4}(p_A, p_B, b) = e^{\widetilde{\mathcal{K}}_{\text{cl}}(p_A, p_B, b)} - 1\,.$$

Natural generalization for spinning particles!

Exponentiation of the classical kernel: an exact solution

- Going to impact parameter space (i.e. to the partial wave basis)

$$\widetilde{f}\,(b) \equiv \int \hat{\mathrm{d}}^4 q \hat{\delta}\,(2p_A \cdot q)\,\hat{\delta}\,(2p_B \cdot q)\, e^{i(q\cdot b)/\hbar} f(q)\,,$$

  the classical BSE becomes

$$\widetilde{\mathcal{M}}^{\mathrm{cl}}_{4,(m+1)}(p_A, p_B, b) = \begin{cases} \widetilde{\mathcal{K}}_{\mathrm{cl}}(p_A, p_B, b) & \text{if } m = 0 \\ \frac{1}{m+1}\widetilde{\mathcal{K}}_{\mathrm{cl}}(p_A, p_B, b)\widetilde{\mathcal{M}}^{\mathrm{cl}}_{4,(m)}(p_A, p_B, b) & \text{if } m \geq 1 \end{cases},$$

  which means that the final solution exponentiates exactly [Adamo,RG]

$$\widetilde{\mathcal{M}}^{\mathrm{cl}}_{4}(p_A, p_B, b) = e^{\widetilde{\mathcal{K}}_{\mathrm{cl}}(p_A, p_B, b)} - 1\,.$$

  Natural generalization for spinning particles!

- The analytic structure (poles, etc.) in momentum space arise completely from

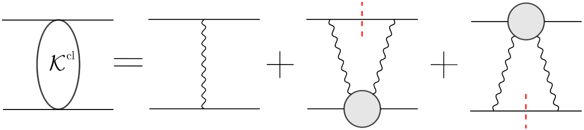
$$\mathrm{i}\mathcal{M}^{\mathrm{cl}}_{4}(p_A, p_B; q_\perp) = \frac{4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}}{\hbar^2} \int \mathrm{d}^2 b\; \mathrm{e}^{-\mathrm{i}\bar{q}_\perp \cdot b} \left(\mathrm{e}^{\widetilde{\mathcal{K}}_{\mathrm{cl}}(p_A, p_B, b)} - 1\right)\,.$$

An example: classical kernel for spinless particles at 2PM

- We can consider for example the classical kernel up to 2 PM

$$\widetilde{\mathcal{K}}^{\mathrm{cl},>}(p_A, p_B, x_\perp) = \frac{i}{\hbar}\Bigg[-2G_N \log(\mu_{\mathrm{IR}}|x_\perp|) m_A m_B \frac{2y^2-1}{\sqrt{y^2-1}} + \frac{3\pi}{4} G_N^2 m_A m_B (m_A + m_B) \frac{5y^2-1}{\sqrt{y^2-1}} \frac{1}{|x_\perp|}\Bigg],$$

which encodes the conservative dynamics of two spinless particles.

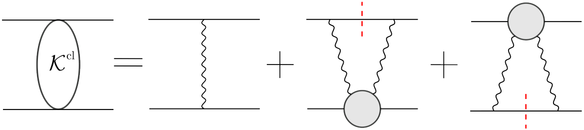


An example: classical kernel for spinless particles at 2PM

- We can consider for example the classical kernel up to 2 PM

$$\widetilde{\mathcal{K}}^{\mathrm{cl},>}(p_A, p_B, x_\perp) = \frac{i}{\hbar}\Bigg[-2G_N \log(\mu_{\mathrm{IR}}|x_\perp|) m_A m_B \frac{2y^2-1}{\sqrt{y^2-1}} + \frac{3\pi}{4} G_N^2 m_A m_B (m_A + m_B) \frac{5y^2-1}{\sqrt{y^2-1}} \frac{1}{|x_\perp|}\Bigg],$$

which encodes the conservative dynamics of two spinless particles.

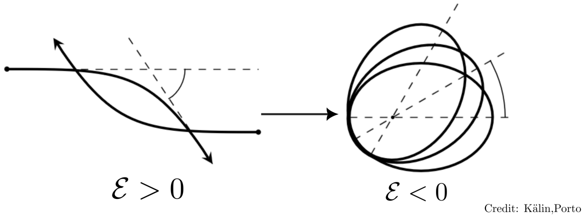


- Note that the motion is restricted to a plane and completely determined by the conserved quantities $(\mathcal{E}, L)$!

$$\mathcal{E} := \frac{E - m_A - m_B}{\mu}, \qquad L = p_\infty(E, m_A, m_B)|x_\perp|, \qquad y = \frac{E^2 - m_A^2 - m_B^2}{2 m_A m_B},$$

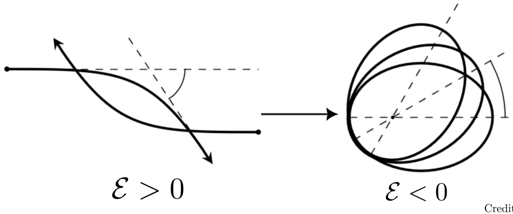
The Hamilton-Jacobi action from amplitudes (I)

- Since $\mathcal{E} > 0$ for scattering orbits and $\mathcal{E} < 0$ for bound orbits, how do we perform an analytic continuation?



The Hamilton-Jacobi action from amplitudes (I)

- Since $\mathcal{E} > 0$ for scattering orbits and $\mathcal{E} < 0$ for bound orbits, how do we perform an analytic continuation?



- Natural connection of the kernel with the scattering Hamilton-Jacobi action

$$\widetilde{\mathcal{K}}_{\rm cl}^{>}(p_A, p_B; x_\perp) = \frac{i}{\hbar} I^{>}(\mathcal{E}, L) \ , \quad I_r^{>}(\mathcal{E}, L) = \oint_{\mathcal{C}^{>}} dr\, p_r(r, \mathcal{E}, L) + L\pi \ ,$$

where p_r is the radial momentum and $\mathcal{C}^{>}$ is the contour of integration for scattering orbits. This is the "amplitude-action" relation! [Bern et al.; Damgaard,Plante,Vanhove; Kol,O'Connell,Telem; Adamo,RG]

The Hamilton-Jacobi action from amplitudes (II)

- There is a remarkable analytic continuation between scattering and bound planar orbits [Kälin,Porto; Adamo, RG, Ilderton]

$$\int_{\mathcal{C}_r^>} = 2\int_{r_m(p_\infty,L)}^{\infty}, \qquad \int_{\mathcal{C}_r^<} = 2\int_{r_-(\tilde{p}_\infty,L)}^{r_+(\tilde{p}_\infty,L)},$$

$$r_-(\tilde{p}_\infty, L) \overset{\mathcal{E}\leq 0}{=} r_m(-i\tilde{p}_\infty, L)\,, \qquad r_+(\tilde{p}_\infty, L) \overset{\mathcal{E}\leq 0}{=} r_m(i\tilde{p}_\infty, L)\,,$$

so that (p_r depends on p_∞^2) [Di Vecchia, Heissenberg, Russo, Veneziano]

$$I_r^<(\tilde{p}_\infty, L) = I_r^>(i\tilde{p}_\infty, L) + I_r^>(-i\tilde{p}_\infty, L)\,.$$

The Hamilton-Jacobi action from amplitudes (II)

- There is a remarkable analytic continuation between scattering and bound planar orbits [Kälin,Porto; Adamo, RG, Ilderton]

$$\int_{\mathcal{C}_r^>} = 2\int_{r_m(p_\infty,L)}^{\infty}\,, \qquad \int_{\mathcal{C}_r^<} = 2\int_{r_-(\tilde{p}_\infty,L)}^{r_+(\tilde{p}_\infty,L)}\,,$$

$$r_-(\tilde{p}_\infty,L) \overset{\mathcal{E}\leq 0}{=} r_m(-i\tilde{p}_\infty,L)\,, \qquad r_+(\tilde{p}_\infty,L) \overset{\mathcal{E}\leq 0}{=} r_m(i\tilde{p}_\infty,L)\,,$$

so that (p_r depends on p_∞^2) [Di Vecchia, Heissenberg, Russo, Veneziano]

$$I_r^<(\tilde{p}_\infty,L) = I_r^>(i\tilde{p}_\infty,L) + I_r^>(-i\tilde{p}_\infty,L)\,.$$

- In the Hamilton-Jacobi picture we can easily compute observables

Scattering angle: $\chi = -\frac{\partial I_r^>}{\partial L}\,,$ Periastron advance: $\Delta\Phi = -\frac{\partial I_r^<}{\partial L}\,.$

The Hamilton-Jacobi action from amplitudes (II)

- There is a remarkable analytic continuation between scattering and bound planar orbits [Kälin,Porto; Adamo, RG, Ilderton]

$$\int_{\mathcal{C}_r^{>}} = 2\int_{r_m(p_\infty,L)}^{\infty}\,, \qquad \int_{\mathcal{C}_r^{<}} = 2\int_{r_-(\tilde{p}_\infty,L)}^{r_+(\tilde{p}_\infty,L)}\,,$$

$$r_-(\tilde{p}_\infty,L) \overset{\mathcal{E}\leq 0}{=} r_m(-i\tilde{p}_\infty,L)\,, \qquad r_+(\tilde{p}_\infty,L) \overset{\mathcal{E}\leq 0}{=} r_m(i\tilde{p}_\infty,L)\,,$$

so that (p_r depends on p_∞^2) [Di Vecchia, Heissenberg, Russo, Veneziano]

$$I_r^{<}(\tilde{p}_\infty,L) = I_r^{>}(i\tilde{p}_\infty,L) + I_r^{>}(-i\tilde{p}_\infty,L)\,.$$

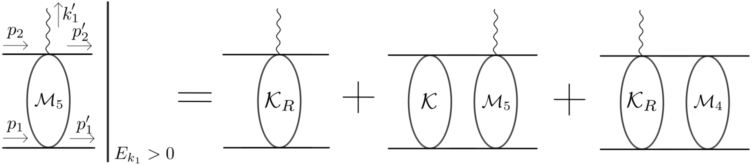
- In the Hamilton-Jacobi picture we can easily compute observables

Scattering angle: $\chi = -\dfrac{\partial I_r^{>}}{\partial L}\,,$ Periastron advance: $\Delta\Phi = -\dfrac{\partial I_r^{<}}{\partial L}\,.$

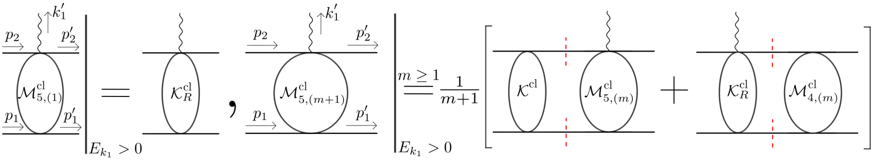
- This picture generalizes for aligned-spin particles $\vec{L}//\vec{a}_1,\vec{a}_2$ [Kälin,Porto], but also for (precessing) generic Kerr orbits [RG, Shi]. How about radiation?

Classical Bethe-Salpeter recursion with radiative effects (I)

- How can the BSE be generalized in the presence of radiation? Consider the 5-pt amplitude recursion with the emission of a positive energy graviton



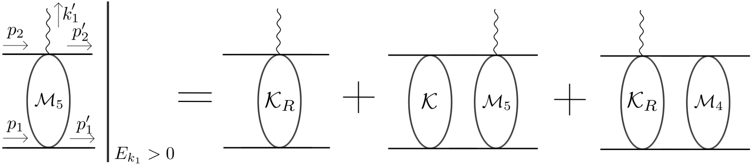
and apply the symmetrization procedure [Adamo, RG, Ilderton]



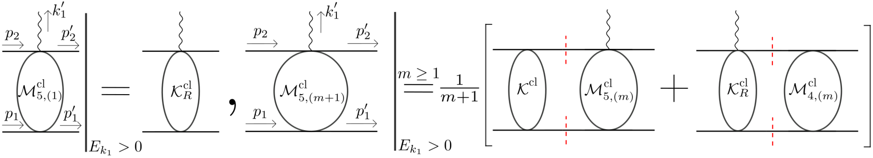
A similar recursion holds for the emission of N gravitons.

Classical Bethe-Salpeter recursion with radiative effects (I)

- How can the BSE be generalized in the presence of radiation? Consider the 5-pt amplitude recursion with the emission of a positive energy graviton



and apply the symmetrization procedure [Adamo, RG, Ilderton]



A similar recursion holds for the emission of *N* gravitons.

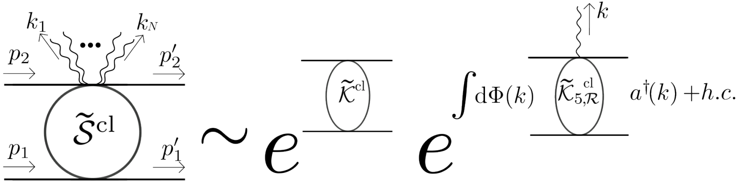
- Can we find an exact solution from the resummation? [Adamo, RG, Ilderton]

Classical Bethe-Salpeter recursion with radiative effects (II)

- The conjectural classical S-matrix is [Cristofoli,RG,Moynihan,O'Connell,Ross, Sergola,White; Britto,RG,Jehu; DiVecchia,Heissenberg,Russo,Veneziano]

$$\widetilde{\mathcal{S}}^{\rm cl}\Big|_{E_{k_1},\ldots,E_{k_N}>0} \sim e^{\widetilde{\mathcal{K}}^{\rm cl}(p_A,p_B;b_1,b_2)} e^{\sum_\sigma \int \mathrm{d}\Phi(k)\widetilde{\mathcal{K}}^{\rm cl}_{5,\mathcal{R}}(p_A,p_B;b_1,b_2,k^\sigma)a^\dagger_\sigma(k)+\text{h.c.}},$$

where a coherent state of gravitons represent the gravitational wave and b_1, b_2 are the impact parameters related to the momentum transfers $q_j = p_j - p'_j$.

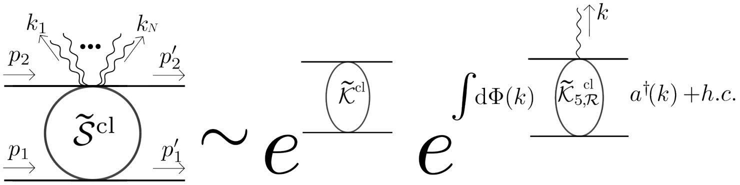


Classical Bethe-Salpeter recursion with radiative effects (II)

- The conjectural classical S-matrix is [Cristofoli,RG,Moynihan,O'Connell,Ross, Sergola,White; Britto,RG,Jehu; DiVecchia,Heissenberg,Russo,Veneziano]

$$\widetilde{\mathcal{S}}^{\mathrm{cl}}\Big|_{E_{k_1},\ldots,E_{k_N}>0} \sim e^{\widetilde{\mathcal{K}}^{\mathrm{cl}}(p_A,p_B;b_1,b_2)} e^{\sum_\sigma \int \mathrm{d}\Phi(k)\widetilde{\mathcal{K}}^{\mathrm{cl}}_{5,\mathcal{R}}(p_A,p_B;b_1,b_2,k^\sigma)a^\dagger_\sigma(k)+\mathrm{h.c.}},$$

  where a coherent state of gravitons represent the gravitational wave and b_1,b_2 are the impact parameters related to the momentum transfers $q_j = p_j - p_j'$.



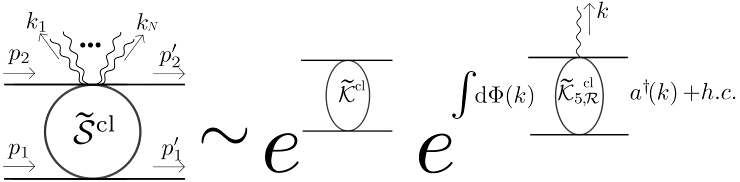
- All scattering and bound observables for the two-body problem can derived from a gauge-invariant representation with 2MPI kernels $\widetilde{\mathcal{K}}^{\mathrm{cl}}$ and $\widetilde{\mathcal{K}}^{\mathrm{cl}}_{5,\mathcal{R}}$!

Classical Bethe-Salpeter recursion with radiative effects (II)

- The conjectural classical S-matrix is [Cristofoli,RG,Moynihan,O'Connell,Ross, Sergola,White; Britto,RG,Jehu; DiVecchia,Heissenberg,Russo,Veneziano]

$$\widetilde{\mathcal{S}}^{\rm cl}\Big|_{E_{k_1},\ldots,E_{k_N}>0} \sim e^{\widetilde{\mathcal{K}}^{\rm cl}(p_A,p_B;b_1,b_2)}\, e^{\sum_\sigma \int \mathrm{d}\Phi(k)\widetilde{\mathcal{K}}^{\rm cl}_{5,\mathcal{R}}(p_A,p_B;b_1,b_2,k^\sigma)a^\dagger_\sigma(k)+\text{h.c.}}\,,$$

where a coherent state of gravitons represent the gravitational wave and b_1,b_2 are the impact parameters related to the momentum transfers $q_j = p_j - p_j'$.



- All scattering and bound observables for the two-body problem can derived from a gauge-invariant representation with 2MPI kernels $\widetilde{\mathcal{K}}^{\rm cl}$ and $\widetilde{\mathcal{K}}^{\rm cl}_{5,\mathcal{R}}$!
- Open problem: can we understand the analytic continuation of the waveform?

PN expansion and time-domain multipoles (I)

- Following the linearized Schwarzschild case, we propose [Adamo,RG,Ilderton]

$$\boxed{h^{<\mathrm{dyn}}(u,\hat{n};\tilde{p}_\infty,L) = h^{>\mathrm{dyn}}(u,\hat{n};p_\infty = +i\tilde{p}_\infty,L)\,, \qquad \mathcal{E}<0\,.}$$

How can this be verified?

PN expansion and time-domain multipoles (I)

- Following the linearized Schwarzschild case, we propose [Adamo,RG,Ilderton]

$$\boxed{h^{<\mathrm{dyn}}(u,\hat{n};\tilde{p}_\infty,L) = h^{>\mathrm{dyn}}(u,\hat{n};p_\infty = +i\tilde{p}_\infty,L)\,, \qquad \mathcal{E}<0\,.}$$

How can this be verified?

- Use the Post-Newtonian expansion: the waveform in the center-of-mass frame is related to the multipole expansion [Bini, Damour, Geralico] in time domain

$$h^{>}\left(u=\frac{b}{p_\infty c}\tilde{u}^{>},\hat{n}\right) = \frac{4G_N}{c^4}\left(W^{>}_{\mathrm{N}} + \frac{1}{c}W^{>}_{0.5\mathrm{PN}} + \frac{1}{c^2}W^{>}_{1\mathrm{PN}} + \dots\right),$$

where the retarded time u needs to be rescaled to obtain the $1/c$ expansion.

PN expansion and time-domain multipoles (I)

- Following the linearized Schwarzschild case, we propose [Adamo,RG,Ilderton]

$$\boxed{h^{<\text{dyn}}(u,\hat{n};\tilde{p}_\infty,L) = h^{>\text{dyn}}(u,\hat{n};p_\infty=+i\tilde{p}_\infty,L)\,, \qquad \mathcal{E}<0\,.}$$

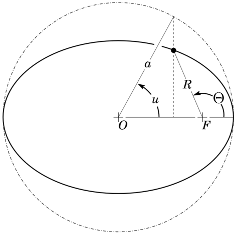
  How can this be verified?

- Use the Post-Newtonian expansion: the waveform in the center-of-mass frame is related to the multipole expansion [Bini, Damour, Geralico] in time domain

$$h^{>}\left(u=\frac{b}{p_\infty c}\tilde{u}^{>},\hat{n}\right) = \frac{4G_N}{c^4}\left(W^{>}_{\text{N}} + \frac{1}{c}W^{>}_{0.5\text{PN}} + \frac{1}{c^2}W^{>}_{1\text{PN}} + \dots\right),$$

  where the retarded time u needs to be rescaled to obtain the $1/c$ expansion.

But PN multipoles can be computed independently with the quasi-Keplerian parametrization for hyperbolic and elliptic orbits! [Damour,Deruelle]



PN expansion and time-domain multipoles (II)

- The scattering and bound (relative) trajectory is $\vec{x} = r(\cos(\phi), \sin(\phi), 0)$

$$r^< = a^<(1 - e_r^< \cos(\mathrm{u}))\,, \qquad r^> = a^>(e_r^> \cosh(\mathrm{v}) - 1)\,,$$

$$n^< t = \mathrm{u} - e_t^< \sin(\mathrm{u}) + \mathcal{O}(1/c)\,, \phi^< = 2k^< \tan^{-1}\left(\sqrt{\frac{e_\phi^< + 1}{1 - e_\phi^<}} \tan\left(\frac{\mathrm{u}}{2}\right)\right) + \mathcal{O}(1/c)\,,$$

$$n^> t = e_t^> \sinh(\mathrm{v}) - \mathrm{v} + \mathcal{O}(1/c)\,, \phi^> = 2k^> \tan^{-1}\left(\sqrt{\frac{e_\phi^> + 1}{e_\phi^> - 1}} \tanh\left(\frac{\mathrm{v}}{2}\right)\right) + \mathcal{O}(1/c)$$

where, analytically continuing in $\mathcal{E}$ up to 1PN, [Damour,Deruelle]

$$n^> \to -in^<\,, e_t^> \to e_t^<\,, e_r^> \to e_r^<\,, e_\phi^> \to e_\phi^<\,, \mathrm{v} \to i\mathrm{u}\,, a^> \to -a^<\,, k^> \to k^<\,.$$

PN expansion and time-domain multipoles (II)

- The scattering and bound (relative) trajectory is $\vec{x} = r(\cos(\phi), \sin(\phi), 0)$

$$r^< = a^<(1 - e_r^< \cos(\mathrm{u}))\,, \qquad r^> = a^>(e_r^> \cosh(\mathrm{v}) - 1)\,,$$

$$n^< t = \mathrm{u} - e_t^< \sin(\mathrm{u}) + \mathcal{O}(1/c)\,, \phi^< = 2k^< \tan^{-1}\left(\sqrt{\frac{e_\phi^< + 1}{1 - e_\phi^<}} \tan\left(\frac{\mathrm{u}}{2}\right)\right) + \mathcal{O}(1/c)\,,$$

$$n^> t = e_t^> \sinh(\mathrm{v}) - \mathrm{v} + \mathcal{O}(1/c)\,, \phi^> = 2k^> \tan^{-1}\left(\sqrt{\frac{e_\phi^> + 1}{e_\phi^> - 1}} \tanh\left(\frac{\mathrm{v}}{2}\right)\right) + \mathcal{O}(1/c)$$

where, analytically continuing in $\mathcal{E}$ up to 1PN, [Damour,Deruelle]

$$n^> \to -in^<\,, e_t^> \to e_t^<\,, e_r^> \to e_r^<\,, e_\phi^> \to e_\phi^<\,, \mathrm{v} \to i\mathrm{u}\,, a^> \to -a^<\,, k^> \to k^<\,.$$

- For the hyperbolic case, to make contact with PM expansion, solve Kepler's equation as an asymptotic expansion at large j to get $\mathrm{v}(t)$

$$\tilde{n}^> t = \frac{1}{jp_\infty}\left[e_t^> \sinh(\mathrm{v}) - \mathrm{v} + \mathcal{O}(1/c)\right]\,, \qquad \tilde{n}_{\mathrm{N}}^> = \frac{n_{\mathrm{N}}^>}{jp_\infty} = \frac{p_\infty c}{b}\,,$$

which gives the relative time-trajectory $\vec{x}(\mathrm{v}(t))$!

PN expansion and time-domain multipoles (III)

- For example, at the Newtonian quadrupole order we now evaluate

$$
\begin{aligned}
W_N^{>}(u) &= \frac{1}{2!}\mathsf{STF}_{ij}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mu x^i(t)x^j(t)\right)\Big|_{t=u} \\
&= -\frac{m_A m_B p_\infty}{4j\left[1+(\tilde{u}^{>})^2\right]^{3/2}}\Big[\left((\tilde{u}^{>})^2+3\right)\cos(2\phi) \\
&\qquad + \left(1+(\tilde{u}^{>})^2\right) + 2\left((\tilde{u}^{>})^3+2\tilde{u}^{>}\right)\sin(2\phi)\Big]
\end{aligned}
$$

which matches the PM tree-level waveform expansion!

PN expansion and time-domain multipoles (III)

- For example, at the Newtonian quadrupole order we now evaluate

$$
\begin{aligned}
W_N^{>}(u) &= \frac{1}{2!}\mathrm{STF}_{ij}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mu x^i(t)x^j(t)\right)\Big|_{t=u} \\
&= -\frac{m_A m_B p_\infty}{4j\left[1+(\tilde{u}^{>})^2\right]^{3/2}}\Big[\left((\tilde{u}^{>})^2+3\right)\cos(2\phi) \\
&\qquad + \left(1+(\tilde{u}^{>})^2\right)+2\left((\tilde{u}^{>})^3+2\tilde{u}^{>}\right)\sin(2\phi)\Big]
\end{aligned}
$$

which matches the PM tree-level waveform expansion!

- In general we find a B2B map between radiative multipoles for hyperbolic and elliptic orbits up to 1PN [Adamo, RG, Ilderton; Junker, Schäfer]

$$
\boxed{W^{<}(u,\tilde{p}_\infty)\Big|_{\mathrm{1PN}} = W^{>}(u,p_\infty=+i\tilde{p}_\infty)\Big|_{\mathrm{1PN}}\,, \qquad \mathcal{E}<0}
$$

which means that our map is independently verified!

Tree-level dynamical waveform for bound orbits

- Using the new B2B map for the waveform, [Adamo,RG,Ilderton]

$$h^{<\mathrm{dyn}}\left(\tilde{u}^{<}\frac{LE}{m_A m_B \tilde{p}_\infty^2 c^2}, \hat{n}\right) = \frac{4G_N}{c^4}\left(W_{\mathrm{N}}^{<\mathrm{dyn}} + \frac{1}{c}W_{0.5\mathrm{PN}}^{<\mathrm{dyn}} + \frac{1}{c^2}W_{1\mathrm{PN}}^{<\mathrm{dyn}} + \dots\right),$$

recovers the PN multipoles $W^{<\mathrm{dyn}}$ computed on the elliptic trajectory.

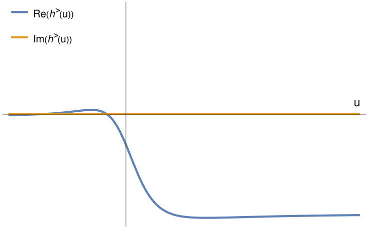
Tree-level dynamical waveform for bound orbits

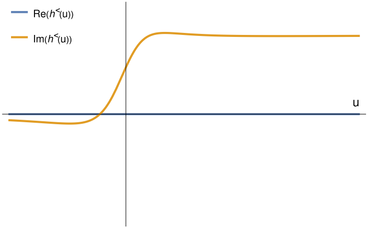
- Using the new B2B map for the waveform, [Adamo,RG,Ilderton]

$$h^{<\mathrm{dyn}}\left(\tilde{u}^{<}\frac{LE}{m_A m_B \tilde{p}_\infty^2 c^2}, \hat{n}\right) = \frac{4G_N}{c^4}\left(W_{\mathrm{N}}^{<\mathrm{dyn}} + \frac{1}{c}W_{0.5\mathrm{PN}}^{<\mathrm{dyn}} + \frac{1}{c^2}W_{1\mathrm{PN}}^{<\mathrm{dyn}} + \dots\right),$$

recovers the PN multipoles $W^{<\mathrm{dyn}}$ computed on the elliptic trajectory.

- We can compare the scattering and bound waveforms in the com frame



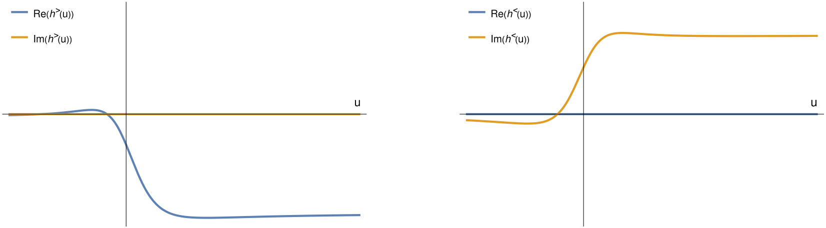


Tree-level dynamical waveform for bound orbits

- Using the new B2B map for the waveform, [Adamo,RG,Ilderton]

$$h^{<\mathrm{dyn}}\left(\tilde{u}^{<}\frac{LE}{m_A m_B \tilde{p}_\infty^2 c^2}, \hat{n}\right) = \frac{4G_N}{c^4}\left(W_{\mathrm{N}}^{<\mathrm{dyn}} + \frac{1}{c}W_{0.5\mathrm{PN}}^{<\mathrm{dyn}} + \frac{1}{c^2}W_{1\mathrm{PN}}^{<\mathrm{dyn}} + \dots\right),$$

  recovers the PN multipoles $W^{<\mathrm{dyn}}$ computed on the elliptic trajectory.
- We can compare the scattering and bound waveforms in the com frame

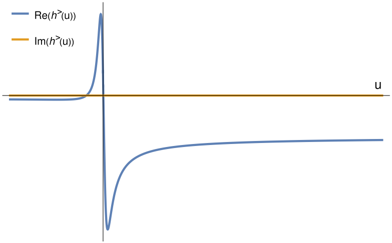


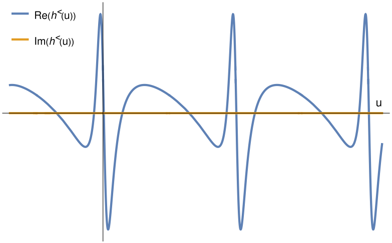
- Why is the bound waveform not periodic in the time u?

From scattering to bound waveforms via resummation

- The analytical continuation of the waveform computed for eccentric orbits requires a resummation in the eccentricity to recover the bound waveform periodicity in the time u [Adamo,RG,Ilderton]

$$n^{<} t = \mathrm{u} - e_t^{<} \sin(\mathrm{u}) + \mathcal{O}\left(1/c\right) \,, \qquad n^{<} t = \mathrm{u} - e_t^{<} \sin(\mathrm{u}) + \mathcal{O}\left(1/c\right) \,.$$

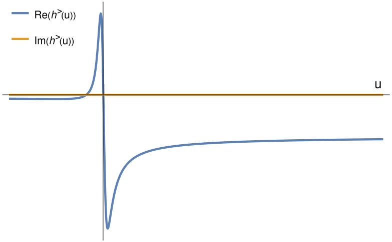


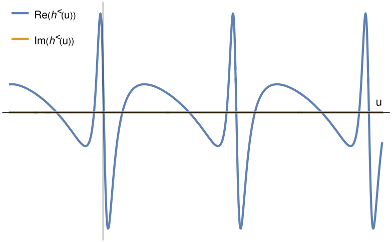


From scattering to bound waveforms via resummation

- The analytical continuation of the waveform computed for eccentric orbits requires a resummation in the eccentricity to recover the bound waveform periodicity in the time u [Adamo,RG,Ilderton]

$$n^{<} t = \mathrm{u} - e_t^{<} \sin(\mathrm{u}) + \mathcal{O}(1/c) \,, \qquad n^{<} t = \mathrm{u} - e_t^{<} \sin(\mathrm{u}) + \mathcal{O}(1/c) \,.$$





- Need to resum perturbative contributions! [WIP with Del Duca, Sasank]

Summary table of the boundary to bound dictionary

- For aligned-spin binaries where the motion remains on the equatorial plane we find a conjectural dictionary [Kälin,Porto;Saketh,Vines,Steinhoff, Buonanno;Cho,Kälin,Porto;Adamo,RG; Heissenberg;Adamo,RG,Ilderton]

Bound observable	Scattering observable
$\Delta\Phi(\tilde{p}_\infty, L, a, c_X)$	$\chi(-i\tilde{p}_\infty, L, a, c_X) + \chi(+i\tilde{p}_\infty, L, a, c_X)$
$\Delta E^{<}_{\text{rad}}(\tilde{p}_\infty, L, a, c_X)$	$\Delta E^{>}_{\text{rad}}(-i\tilde{p}_\infty, L, a, c_X) + \Delta E^{>}_{\text{rad}}(+i\tilde{p}_\infty, L, a, c_X)$
$\Delta J^{<}_{\text{rad}}(\tilde{p}_\infty, L, a, c_X)$	$\Delta J^{>}_{\text{rad}}(-i\tilde{p}_\infty, L, a, c_X) + \Delta J^{>}_{\text{rad}}(+i\tilde{p}_\infty, L, a, c_X)$

which is valid at least up to 3PM (G_N^3) order for the scattering angle $\Delta\chi$/periastron advance $\Delta\Phi$ and for the fluxes $\Delta E_{\text{rad}}, \Delta J_{\text{rad}}$.

Summary table of the boundary to bound dictionary

- For aligned-spin binaries where the motion remains on the equatorial plane we find a conjectural dictionary [Kälin,Porto;Saketh,Vines,Steinhoff, Buonanno;Cho,Kälin,Porto;Adamo,RG; Heissenberg;Adamo,RG,Ilderton]

Bound observable	Scattering observable
$\Delta\Phi(\tilde{p}_\infty, L, a, c_X)$	$\chi(-i\tilde{p}_\infty, L, a, c_X) + \chi(+i\tilde{p}_\infty, L, a, c_X)$
$\Delta E^{<}_{\rm rad}(\tilde{p}_\infty, L, a, c_X)$	$\Delta E^{>}_{\rm rad}(-i\tilde{p}_\infty, L, a, c_X) + \Delta E^{>}_{\rm rad}(+i\tilde{p}_\infty, L, a, c_X)$
$\Delta J^{<}_{\rm rad}(\tilde{p}_\infty, L, a, c_X)$	$\Delta J^{>}_{\rm rad}(-i\tilde{p}_\infty, L, a, c_X) + \Delta J^{>}_{\rm rad}(+i\tilde{p}_\infty, L, a, c_X)$

  which is valid at least up to 3PM (G_N^3) order for the scattering angle $\Delta\chi$/periastron advance $\Delta\Phi$ and for the fluxes $\Delta E_{\rm rad}, \Delta J_{\rm rad}$.
- New waveform map (up to 1PN and tree-level)[Adamo,RG,Ilderton]

$$\boxed{h^{<\rm dyn}(u; \tilde{p}_\infty, L, a, c_X) = h^{>\rm dyn}(u; +i\tilde{p}_\infty, L, a, c_X)}$$

  in agreement with the prescription for the orbital elements [Damour,Deruelle]

Summary table of the boundary to bound dictionary

- For aligned-spin binaries where the motion remains on the equatorial plane we find a conjectural dictionary [Kälin,Porto;Saketh,Vines,Steinhoff, Buonanno;Cho,Kälin,Porto;Adamo,RG; Heissenberg;Adamo,RG,Ilderton]

Bound observable	Scattering observable
$\Delta\Phi(\tilde{p}_\infty, L, a, c_X)$	$\chi(-i\tilde{p}_\infty, L, a, c_X) + \chi(+i\tilde{p}_\infty, L, a, c_X)$
$\Delta E^{<}_{\text{rad}}(\tilde{p}_\infty, L, a, c_X)$	$\Delta E^{>}_{\text{rad}}(-i\tilde{p}_\infty, L, a, c_X) + \Delta E^{>}_{\text{rad}}(+i\tilde{p}_\infty, L, a, c_X)$
$\Delta J^{<}_{\text{rad}}(\tilde{p}_\infty, L, a, c_X)$	$\Delta J^{>}_{\text{rad}}(-i\tilde{p}_\infty, L, a, c_X) + \Delta J^{>}_{\text{rad}}(+i\tilde{p}_\infty, L, a, c_X)$

  which is valid at least up to 3PM (G_N^3) order for the scattering angle $\Delta\chi$/periastron advance $\Delta\Phi$ and for the fluxes $\Delta E_{\text{rad}}, \Delta J_{\text{rad}}$.

- New waveform map (up to 1PN and tree-level)[Adamo,RG,Ilderton]

$$h^{<\text{dyn}}(u; \tilde{p}_\infty, L, a, c_X) = h^{>\text{dyn}}(u; +i\tilde{p}_\infty, L, a, c_X)$$

  in agreement with the prescription for the orbital elements [Damour,Deruelle]

- Need to study tail effects appearing at higher orders! [Cho,Kälin,Porto]

Summary and future directions

- We extend the KMOC formalism to study the scattering waveform relevant for the classical two-body problem using EFT and amplitude techniques

Summary and future directions

- We extend the KMOC formalism to study the scattering waveform relevant for the classical two-body problem using EFT and amplitude techniques
- We showed how the analitic properties of the S-matrix allow to derive a universal expression of the leading order time-domain waveform in terms of the 3-pt and 4-pt Compton amplitude, bypassing completely the integration of the 5-pt amplitude

Summary and future directions

- We extend the KMOC formalism to study the scattering waveform relevant for the classical two-body problem using EFT and amplitude techniques
- We showed how the analitic properties of the S-matrix allow to derive a universal expression of the leading order time-domain waveform in terms of the 3-pt and 4-pt Compton amplitude, bypassing completely the integration of the 5-pt amplitude
- We discussed a generalization of the boundary to bound dictionary for various observables, including the post-Minkowskian waveform, matching the result with the calculation of the radiative multipoles in post-Newtonian limit and making contact with the analytic continuation of the orbital elements

Summary and future directions

- We extend the KMOC formalism to study the scattering waveform relevant for the classical two-body problem using EFT and amplitude techniques
- We showed how the analitic properties of the S-matrix allow to derive a universal expression of the leading order time-domain waveform in terms of the 3-pt and 4-pt Compton amplitude, bypassing completely the integration of the 5-pt amplitude
- We discussed a generalization of the boundary to bound dictionary for various observables, including the post-Minkowskian waveform, matching the result with the calculation of the radiative multipoles in post-Newtonian limit and making contact with the analytic continuation of the orbital elements
- We emphasize the need to resum perturbative contributions for the bound waveform to make contact with phenomenological applications

Summary and future directions

- We extend the KMOC formalism to study the scattering waveform relevant for the classical two-body problem using EFT and amplitude techniques
- We showed how the analitic properties of the S-matrix allow to derive a universal expression of the leading order time-domain waveform in terms of the 3-pt and 4-pt Compton amplitude, bypassing completely the integration of the 5-pt amplitude
- We discussed a generalization of the boundary to bound dictionary for various observables, including the post-Minkowskian waveform, matching the result with the calculation of the radiative multipoles in post-Newtonian limit and making contact with the analytic continuation of the orbital elements
- We emphasize the need to resum perturbative contributions for the bound waveform to make contact with phenomenological applications
- Future directions: extend the scattering-to-bound map to include tail effects for the waveform and other observables, explore the resummation of PM contributions, extend the scattering-to-bound map to generic spins, . . .

Summary and future directions